Geodesic flows for the Neumann-Rosochatius systems

REIJIRO KUBO†, WAICHI OGURA‡, TAKESI SAITO§ AND YUKINORI YASUI‖

†*Yukawa Institute for Theoretical Physics Kyoto University, Kyoto 606, Japan*
‡*Department of Physics, Osaka University, Toyonaka, Osaka 565, Japan*
§*Department of Physics, Kwansei Gakuin University, Nishinomiya 662, Japan*
‖*Department of Physics, Osaka City University, Sumiyoshiku, Osaka 558, Japan*

ABSTRACT

The Relationship between the Neumann system and the Jacobi system in arbitrary dimensions is elucidated from the point of view of constrained Hamiltonian systems. Dirac brackets for canonical variables of both systems are derived from the constrained Hamiltonians. The geodesic equations corresponding to the Rosochatius system are studied as an application of our method. As a consequence a new class of nonlinear integrable equations is derived along with their conserved quantities.

1. Introduction

To deal with Hamiltonian integrable systems, we usually rely on some basic procedures such as determination of action-angle variables, separation of variables, and integration of equations of motion. In addition, it is often very instructive to study relations which exist among some superficially different mechanical systems.

For instance, it is known that there is a close relationship between the Neumann system [1] and the Jacobi system [2]. The Neumann system describes the motion of a point particle on the $N-1$ sphere, S^{N-1}, under the influence of a quadratic potential in N-dimensional space. Although the Neumann system has been known for more than 100 years as an integrable nonlinear mechanical system, it is still under active study [3-7]. On the other hand, the Jacobi system describes the geodesic flow on an ellipsoid. At first sight these two mechanical systems are different. However, Knörrer [7] found the mapping from the Jacobi system onto the Neumann system, showing that the two systems are essentially equivalent.

A more complicated problem was recently studied by Adler and van Moerbeke [8]. They found that there are rational maps transforming the Kowalewski top and the Hénon-Heiles system into Manakov's geodesic flow on $SO(4)$.

In the present work we deal with some classical integrable systems such as the Jacobi, Neumann, and Rosochatius systems laying special emphasis on the relationship which exists among them. In sections 2 and 3, we deal with the Jacobi and the Neumann systems in the framework of the constrained Hamilton formalism with a set of second-class constraints. We calculate the Dirac brackets for canonical variables in the Jacobi and the Neumann systems. We then transform variables so as to reduce second-class constraints to first-class ones. As a consequence, both systems acquire gauge freedoms.

The relationship between the Neumann system and the Jacobi system is clarified by making use of the Gauss map introduced by Knörrer[7]. As an application of our method, we then study in section 4 the geodesic equations corresponding to the Rosochatius system [9,10], which is also known as an integrable nonlinear

system. Hamilton's equations for the N-dimensional Rosochatius system can be obtained from the $2N$-dimensional Neumann system. By the same token, we found equations corresponding to the geodesic equations for the Rosochatius system from those of the $2N$-dimensional Jacobi system. We show that the equations obtained are integrable by finding integrals of motion explicitly. The last section is devoted to summary and concluding remarks.

2. The Neumann system

In this section we deal with the famous Neumann system [1] in arbitrary dimensions in the framework of the constrained Hamiltonian formalism with second-class constraints. Recently,various aspects of the Neumann system are investigated by many authors [3-7], because the Neumann system is interesting as one of the simplest integrable non-linear systems and it is closely connected to other non-linear mechanical systems, such as those of the KdV, the Kowalewski top etc.. Our aim here is to clarify the Poisson structure of the Neumann system.

Let $(x_1, ..., x_N, v_1, ..., v_N)$ be canonical coordinates, so that the Poisson brackets are given by $\{x_i,\ v_j\}_P = \delta_{ij}, \{x_i,\ x_j\}_P = \{v_i,\ v_j\}_P = 0$, and the symplectic form ω is written as $\omega = \sum_{k=1}^{N} dx_k \wedge dv_k$. The Hamiltonian for the Neumann system is written as

$$\mathcal{H} = \mathcal{H}_N + \xi\ \Omega_1 + \eta\ \Omega_2, \tag{1}$$

where ξ and η are Lagrange multipliers for the second-class constraints

$$\begin{aligned} \Omega_1 &= \frac{1}{2}\Big(\sum_{k=1}^{N} x_k^2 - r^2\Big), \\ \Omega_2 &= \sum_{k=1}^{N} x_k v_k \end{aligned} \tag{2}$$

3

and the classical Hamiltonian $\mathcal{H}_N$ for the Neumann system is given by

$$\mathcal{H}_N = \frac{1}{2} \sum_{k=1}^{N} (v_k^2 + a_k x_k^2) \tag{3}$$

with $r(\neq 0)$ and $0 < a_1 < a_2 < \cdots < a_N$ being real constant parameters.

The Hamilton equations derived from (1) are

$$\dot{x}_k - v_k - \eta x_k = 0, \tag{4}$$

$$\dot{v}_k + a_k x_k + \xi x_k + \eta v_k = 0, \tag{5}$$

$$\sum_{k=1}^{N} x_k^2 - r^2 = 0, \tag{6}$$

$$\sum_{k=1}^{N} x_k v_k = 0. \tag{7}$$

It follows from (4) and (6) that

$$\sum_{k=1}^{N} x_k \dot{x}_k = \sum_{k=1}^{N} x_k v_k + \eta \ r^2 = 0, \tag{8}$$

so that

$$\eta = 0. \tag{9}$$

Also we have from (4),(5) and (7)

$$\sum_{k=1}^{N} (x_k \dot{v}_k + v_k \dot{x}_k) = \sum_{k=1}^{N} (v_k^2 - a_k x_k^2) - \xi \ r^2 = 0. \tag{10}$$

It follows from (4),(5) and (9) that

$$\ddot{x}_k = \dot{v}_k = -a_k x_k - \xi x_k. \tag{11}$$

4

Combining (10) with (11) one obtains

$$\ddot{x}_k = -a_k x_k - \frac{x_k}{r^2} \sum_{j=1}^{N} (\dot{x}_j^2 - a_j x_j^2). \qquad (12)$$

The Dirac brackets [12,13] evaluated from (1) are given by

$$\{x_i, \ x_j\}_D = 0, \qquad (13)$$

$$\{x_i, \ v_j\}_D = \delta_{ij} - \frac{x_i x_j}{x^2}, \qquad (14)$$

$$\{v_i, \ v_j\}_D = -\frac{(x_i v_j - x_j v_i)}{x^2}, \qquad (15)$$

where $x^2 \equiv \sum_{k=1}^{N} x_k^2$.

Let us now introduce new variables $y_i, \ (i = 1, ...N)$, defined by

$$v_i = y_i - \frac{x_i}{x^2}(x \cdot y), \qquad (16)$$

where $x \cdot y \equiv \sum_{k=1}^{N} x_k y_k$. By this change of variables, one obtains the identity

$$x \cdot v \equiv \sum_{k=1}^{N} x_k v_k = 0. \qquad (17)$$

Therefore, the term proportional to η in the Hamiltonian (1) vanishes identically. We see that (7) also turns out to be an identity. It should be noted here that the change of variables gives rise to a qualitative change of the class of constraints. Namely, although Ω_1 and Ω_2 were originally second-class constraints, after the change of variables Ω_2 vanishes identically and Ω_1 turns out to be a first-class constraint with the nonvanishing Lagrange-multiplier ξ. As a consequence, the system acquires gauge freedom and ξ becomes the gauge parameter.

In order to see the gauge freedom explicitly, let us introduce the Lagrangian $\mathcal{L}$ for the Neumann system by the Legendre transform

$$\mathcal{L} = \sum_{k=1}^{N} \dot{x}_k v_k - \mathcal{H}. \tag{18}$$

Setting $\Omega_2 = 0$ and substituting (16) into (18), we obtain

$$\mathcal{L}_{(\Omega_2=0)} = \sum_{k=1}^{N} \dot{x}_k (y_k + \lambda x_k) - \frac{1}{2x^2} \sum_{k>l}^{N} J_{kl}^2 - \frac{1}{2} \sum_{k=1}^{N} a_k x_k^2, \tag{19}$$

where $\lambda = \zeta - ((x \cdot y)/x^2)$, $\dot{\zeta} = \xi$, $J_{kl} = x_k y_l - x_l y_k$, and total derivative terms with respect to time have been deleted from (19).

The Euler equations obtained from (19) are

$$\dot{x}_k = y_k - \frac{(x \cdot y)}{x^2} x_k, \tag{20}$$

$$\sum_{k=1}^{N} x_k \dot{x}_k = 0, \tag{21}$$

$$\dot{y}_k = -\dot{\lambda} x_k - a_k x_k + \frac{1}{x^2}(x \cdot y) y_k. \tag{22}$$

It follows from (20) and (22) that

$$\ddot{x}_k = -a_k x_k - \frac{x_k}{x^2} \sum_{l=1}^{N} (\dot{x}_l^2 - a_l x_l^2), \tag{23}$$

where we have used the identity

$$\sum_{k=1}^{N} \dot{x}_k^2 = y^2 - \frac{(x \cdot y)^2}{x^2}. \tag{24}$$

Eq.(21) implies that

$$x^2 = \sum_{k=1}^{N} x_k^2 = \text{const.}. \tag{25}$$

On the other hand, changing variables from v_k to y_k by (16), we have from

(13),(14) and (15) that

$$\{x_i,\ x_j\}_P = 0, \tag{26}$$

$$\{x_i,\ y_j\}_P = \delta_{ij}, \tag{27}$$

$$\{y_i,\ y_j\}_P = -\frac{(x_i y_j - x_j y_i)}{x^2}, \tag{28}$$

It should be noted that the Dirac brackets (13),(14),and (15) turn out to be the simple Poisson brackets here.

Since J_{kl} is invariant under the gauge transformation,

$$(x_k,\ y_k) \Longrightarrow (x_k,\ y_k + \lambda x_k), \tag{29}$$

where λ is any function of x and y, the Hamiltonian

$$\mathcal{H}_N = \frac{1}{2x^2} \sum_{k>l}^{N} J_{kl}^2 + \frac{1}{2} \sum_{k=1}^{N} a_k x_k^2 \tag{30}$$

is also invariant under the gauge transformation (29), and so does the equation of motion (23). Although the parameter λ in (19) plays a role of a Lagrange multiplier in dealing with the constraint $\sum_{k=1}^{N} x_k \dot{x}_k = 0$ imposed on the system, it can be regarded, on the other hand, as a gauge parameter. One notices that the change of variables from v_k to y_k, $(k = 1, ..., N)$, gives rise to a transmutation of constraints from the second-class to the first-class ones, so that a gauge freedom appears through the gauge parameter λ in the system.

The Neumann system is Liouville integrable, because there exist $N - 1$ independent quantities [10], which are constants of motion and which are in involution. In fact define the gauge-invariant quantities[11]

$$F_k = x_k^2 + \frac{1}{x^2} \sum_{l \neq k}^{N} \frac{J_{kl}^2}{a_k - a_l}. \tag{31}$$

Then, we can easily show that the F_k 's are conserved quantities, $\dot{F}_k = 0$, and that

they are in involution

$$\{F_k,\ F_l\}_P = 0, \quad (k, l = 1, ..., N). \tag{32}$$

$N - 1$ of them are independent, because

$$\sum_{k=1}^{N} F_k = x^2 = \text{const.}. \tag{33}$$

The Hamiltonian for the Neumann system is expressed in terms of the $F_k\,'s$ as

$$\mathcal{H}_N = \frac{1}{2} \sum_{k=1}^{N} a_k F_k. \tag{34}$$

3. The Jacobi system

The Neumann system is closely related to the mechanical system of the geodesic motion of a particle on an ellipsoid, which was already shown to be integrable by Jacobi in 1838 [2]. The Lagrangian of the geodesic motion is written as $\mathcal{L} = \sum_{k=1}^{N} q_k' p_k - \mathcal{H}$, where the Hamiltonian $\mathcal{H}$ of the geodesic motion is given by

$$\mathcal{H} = \mathcal{H}_J + \frac{\xi}{2}\left(\sum_{k=1}^{N} \frac{q_k^2}{b_k} - 1\right) + \eta \sum_{k=1}^{N} \frac{q_k p_k}{b_k} \tag{35}$$

with $\mathcal{H}_J = \frac{1}{2}p^2 = \frac{1}{2}\sum_{k=1}^{N} p_k^2$. The prime " $'$ " denotes the differentiation with respect to the parameter s, e.g. $q_k' = dq_k/ds$. The Hamilton equations derived from (35) are

$$q_k' - p_k - \eta\frac{q_k}{b_k} = 0, \tag{36}$$

$$p_k' + \xi\frac{q_k}{b_k} + \eta\frac{p_k}{b_k} = 0, \tag{37}$$

8

$$\sum_{k=1}^{N} \frac{q_k^2}{b_k} - 1 = 0, \tag{38}$$

$$\sum_{k=1}^{N} \frac{q_k p_k}{b_k} = 0. \tag{39}$$

It follows from (36),(38) and (39) that

$$\sum_{k=1}^{N} \frac{q_k q_k'}{b_k} = \sum_{k=1}^{N} \frac{q_k p_k}{b_k} + \eta \sum_{k=1}^{N} \frac{q_k^2}{b_k^2} = 0, \tag{40}$$

so that

$$\eta = 0. \tag{41}$$

Also we have from (36),(37) and (39)

$$\sum_{k=1}^{N} \frac{q_k p_k' + p_k q_k'}{b_k} = \sum_{k=1}^{N} \frac{p_k^2}{b_k} - \xi \sum_{k=1}^{N} \frac{q_k^2}{b_k^2} = 0. \tag{42}$$

Thus we have

$$q_k'' = p_k' = -\xi \frac{q_k}{b_k} = -\frac{1}{R^2} \left(\sum_{l=1}^{N} \frac{p_l^2}{b_l} \right) \frac{q_k}{b_k}, \tag{43}$$

where

$$R^2 \equiv \sum_{l=1}^{N} (q_l^2/b_l^2). \tag{44}$$

The Dirac brackets are found to be

$$\{q_i, \ q_j\}_D = 0, \tag{45}$$

$$\{q_i, \ p_j\}_D = \delta_{ij} - \frac{1}{R^2} \frac{q_i q_j}{b_i b_j}, \tag{46}$$

$$\{p_i, \ p_j\}_D = -\frac{1}{R^2} \frac{(q_i p_j - q_j p_i)}{b_i b_j}. \tag{47}$$

9

In order to see the connection between the Neumann and the Jacobi systems, let us introduce new variables $x_i, (i = 1, ..., N)$ and the time variable t by [7]

$$x_i = \frac{r}{R} \frac{q_i}{b_i},$$
(48)

$$\frac{ds}{dt} = \kappa R^2.$$
(49)

Here $\kappa^2 = (R^2 \sum_{l=1}^{N} p_l^2/b_l)^{-1}$ can be shown to be a constant. Therefore, we have the relation

$$\frac{1}{R^2} \left(\sum_{l=1}^{N} \frac{p_l^2}{b_l} \right) \dot{s}^2 = 1,$$
(50)

where $\dot{s} = ds/dt$. The nonlinear mapping from q to x by (48) is a kind of the Gauss mapping, which maps the ellipsoid onto the sphere with the radius r.

Differentiating both sides of (48) with respect to t twice, we obtain

$$\ddot{x}_k = -\frac{r}{R^3} \left(\sum_{l=1}^{N} \frac{p_l^2}{b_l} \right) \dot{s}^2 \frac{q_k}{b_k^2} + \frac{r}{R} \left(\ddot{s} - \frac{2R'}{R} \dot{s}^2 \right) \frac{q_k'}{b_k}$$
$$- \frac{r}{R^2} \left(R'' \dot{s}^2 - \frac{2(R')^2}{R} \dot{s}^2 + R' \ddot{s} \right) \frac{q_k}{b_k}.$$
(51)

Also differentiating both sides of (49) with respect to t one obtains

$$\ddot{s} = \frac{2R'}{R} \dot{s}^2.$$
(52)

Substitution of (49), (50) and (52) into (51) gives

$$\ddot{x}_k = -\frac{x_k}{b_k} - RR'' \left(\sum_{l=1}^{N} \frac{p_l^2}{b_l} \right)^{-1} x_k$$
$$= -\frac{x_k}{b_k} - \frac{x_k}{r^2} \sum_{l=1}^{N} (\dot{x}_l^2 - \frac{x_l^2}{b_l}).$$
(53)

10

Putting $1/b_k = a_k$ and $r^2 = x^2$ we finally arrive at Neumann's equations again:

$$\ddot{x}_k = -a_k x_k - \frac{x_k}{x^2} \sum_{l=1}^{N} (\dot{x}_l^2 - a_l x_l^2).$$ (54)

It should be noted that the first derivative of x_k can be written as

$$\dot{x}_k = y_k - \frac{(x \cdot y)}{x^2} x_k,$$ (55)

where

$$y_k = \frac{r}{R} \frac{p_k}{b_k} \dot{s} + \mu x_k = \frac{r}{Rb_k} (p_k \dot{s} + \mu q_k)$$ (56)

with μ an arbitrary gauge parameter, which may be a function of x and y. Eq.(55) is corresponding to (20). Differentiating (56) with respect to t, one finds

$$\dot{y}_k = -\frac{1}{b_k} x_k + \frac{(x \cdot y)}{x^2} y_k - 2\mu \frac{(x \cdot y)}{x^2} x_k + \mu^2 x_k + \dot{\mu} x_k$$ (57)

This equation agrees with (22), if one fixes the gauge in (22) as follows:

$$\dot{\lambda} = \frac{y^2}{x^2} - 2\mu \frac{(x \cdot y)}{x^2} + \mu^2 + \dot{\mu}$$ (58)

with $a_k = 1/b_k$. Needless to say that Neumann's equations also follow from (55) and (57).

It is to be noted that the Jacobi system is also Liouville integrable, because there exist N gauge-invariant quantities G_k, $(k = 1, ..., N)$, corresponding to F_k in (31) [11],

$$G_k = p_k^2 + \sum_{l \neq k}^{N} \frac{K_{kl}^2}{b_k - b_l}$$ (59)

with

$$K_{kl} = q_k p_l - q_l p_k,$$ (60)

11

which are also shown to be constants of motion, $G_k' = 0$, and

$$\{G_k,\ G_l\}_P = 0. \tag{61}$$

$N - 1$ of them are independent, because

$$\sum_{k=1}^{N} \frac{G_k}{b_k} = \left(1 - \sum_{k=1}^{N} \frac{q_k^2}{b_k}\right) \sum_{k=1}^{N} \frac{p_k^2}{b_k} - \left(\sum_{k=1}^{N} \frac{q_k p_k}{b_k}\right)^2 = 0. \tag{62}$$

Hamiltonian for the Jacobi system can be written in terms of G_k as follows:

$$\mathcal{H}_J = \frac{1}{2} \sum_{k=1}^{N} G_k = \frac{1}{2} p^2. \tag{63}$$

One finds from (31), (48), (56) and (59) the relation

$$F_k = \frac{r^2 \dot{s}^2}{R^4 b_k^2} (p_k^2 - G_k) = -\frac{r^2 \kappa^2}{b_k^2} \sum_{l \neq k}^{N} \frac{K_{kl}^2}{b_k - b_l}, \tag{64}$$

from which one obtains useful formulas such as

$$\sum_{k=1}^{N} b_k F_k = \sum_{k=1}^{N} \frac{F_k}{a_k} = \frac{r^2}{R^2}, \tag{65}$$

$$\sum_{k=1}^{N} b_k^2 F_k = \sum_{k=1}^{N} \frac{F_k}{a_k^2} = 0. \tag{66}$$

It also follows from (50) and (56) that

$$\sum_{k=1}^{N} \frac{y_k^2}{a_k} = r^2 \left(1 + \frac{\mu^2}{R^2}\right). \tag{67}$$

It is interesting to note that this equation is transformed to

$$\sum_{k=1}^{N} \frac{z_k^2}{a_k} = r^2 \left(1 + \frac{(\mu + \nu)^2}{R^2}\right) \tag{68}$$

under the gauge transformation $y_k \Longrightarrow z_k = y_k + \nu x_k$. In consequence, the phase space of the Jacobi system is mapped by (48) and (56) onto the sphere defined by

12

$x^2 = r^2$ and the manifold given by (67) in the phase space of the Neumann system. If one fixes the gauge parameter μ, the mapping is obviously bijective.

4. The Rosochatius system

In this section we generalize the formalism developed in preceding sections and consider the Rosochatius system [9,10] in the generalized framework. Let us start with rewriting (48) in the form

$$x_i = \frac{r}{R} f_i(q), \tag{71}$$

where $f_i(q)$, $(i = 1, ..., N)$, are regular functions of the coordinates q_j, $(j = 1, ..., N)$ which correspond to those introduced in the Jacobi system. The function $R = R(q)$ is assumed to be given in terms of $f_i(q)$ as follows:

$$R^2 = \sum_{i=1}^{N} f_i^2(q) \tag{72}$$

so that

$$\sum_{i=1}^{N} x_i^2 = \frac{r^2}{R^2} \sum_{i=1}^{N} f_i^2 = r^2 = \text{const.} \tag{73}$$

showing that x_i, $(i = 1, ..., N)$ are variables on the $N - 1$ sphere S^{N-1}.

Differentiating x_i with respect to t, one obtains

$$\dot{x}_i = y_i - \frac{(xy)}{x^2} x_i, \tag{74}$$

where

$$y_i = \frac{r}{R} \dot{s} \, f_i' + \mu x_i \tag{75}$$

with μ the gauge parameter, $\dot{s} = ds/dt$ and $f_i' = df_i/ds$.

13

Differentiating both sides of (74) once again with respect to t, one obtains

$$\ddot{x}_i = \dot{y}_i - \frac{x_i}{x^2}(\dot{x}^2 + x\dot{y}) - \frac{(xy)}{x^2}y_i + \frac{(xy)^2}{x^4}x_i. \tag{76}$$

As we noticed in the case of the Neumann system, $\dot{y}_i$ is written in terms of x_i and y_i as follows:

$$\dot{y}_i = A_i x_i + B y_i, \tag{77}$$

where

$$B = \frac{(xy)}{x^2}, \tag{78}$$
$$A_i = -a_i, \qquad (i = 1, ..., N).$$

If we take

$$A_i = -a_i + \frac{c_i}{x_i^4} \tag{79}$$

in addition to $B - (xy)/x^2 = 0$, then (76) turns out to be

$$\ddot{x}_i = -a_i x_i + \frac{c_i}{x_i^3} - \frac{x_i}{x^2}\left(\dot{x}^2 - \sum_{k=1}^N a_k x_k^2 + \sum_{k=1}^N \frac{c_k}{x_k^2}\right), \tag{80}$$

which are the equations of motion of a particle on the sphere S^{N-1} under the influence of the potential

$$U(x) = \frac{1}{2}\sum_{k=1}^N \left(a_k x_k^2 + \frac{c_k}{x_k^2}\right). \tag{81}$$

Eqs.(80) are known as the Rosochatius equations, and the Hamiltonian system governed by the potential $U(x)$ given above is called the Rosochatius system [9,10]. This system has been shown to be completely integrable by Rosochatius.

14

The geodesic equations (43) in the Jacobi system is rewritten in terms of the f_i's as

$$\frac{d^2 f_i}{ds^2} + \frac{1}{R^2} a_i f_i \sum_{j=1}^{N} \frac{1}{a_j} \left(\frac{df_j}{ds}\right)^2 = 0. \tag{82}$$

Our final task is to find the geodesic equations for the Rosochatius system corresponding to (82) in the Jacobi system. In order to do this we first consider the $2N$-dimensional Neumann system given by the Hamiltonian

$$\mathcal{H} = \mathcal{H}_{2N} + \frac{1}{2}\xi\left(\sum_{i=1}^{2N} z_i^2 - r^2\right) + \eta \sum_{j=1}^{2N} z_j w_j, \tag{83}$$

where

$$\mathcal{H}_{2N} = \frac{1}{2}\sum_{k=1}^{N} [w_k^2 + w_{k+N}^2 + a_k z_k^2 + a_{k+N} z_{k+N}^2] \tag{84}$$

with $w_i = \dot{z}_i$ $(i = 1, ..., 2N)$ and $a_{k+N} = a_k$ [3].

We introduce new variables x_k and θ_k, $(k = 1, ..., N)$, by

$$z_k = x_k \cos\theta_k, \qquad z_{k+N} = x_k \sin\theta_k \tag{85}$$

and rewrite $\mathcal{H}_{2N}$ in terms of the new variables as

$$\mathcal{H}_{2N} = \frac{1}{2}\sum_{k=1}^{N}(\dot{x}_k^2 + x_k^2\dot{\theta}_k^2 + a_k x_k^2). \tag{86}$$

We then restrict ourselves to the specific case in which the angular momenta $z_k w_{k+N} - z_{k+N} w_k$, $(k = 1, ..., N)$ are the constants of motion, that is,

$$z_k w_{k+N} - z_{k+N} w_k = x_k^2 \dot{\theta}_k = \sqrt{c_k} \ (k = 1, ..., N) \tag{87}$$

with c_k, $(k = 1, ..., N)$ real constants. Substituting (87) into (86), we obtain the

15

Hamiltonian for the Rosochatius system

$$\mathcal{H}_R = \frac{1}{2}\sum_{k=1}^{N}(\dot{x}_k^2 + \frac{c_k}{x_k^2} + a_k x_k^2). \tag{88}$$

We next consider the $2N$-dimensional Jacobi system. The Hamiltonian is given by (35) with N replaced by $2N$ and $b_k = b_{k+N} = 1/a_k, (k = 1, ..., N)$. The geodesic equations are then written as

$$\frac{d^2 g_i}{ds^2} + \frac{1}{R^2}a_i g_i \sum_{j=1}^{N}\frac{1}{a_j}\left(\left(\frac{dg_j}{ds}\right)^2 + \left(\frac{dg_{j+N}}{ds}\right)^2\right) = 0, \tag{89}$$

where

$$z_j = \frac{r}{R}g_j, \qquad z_{j+N} = \frac{r}{R}g_{j+N}. \tag{90}$$

In view of (85) it is convenient to introduce the variables f_i, $(i = 1, ..., N)$ by

$$g_i = f_i \cos\theta_i, \qquad g_{i+N} = f_i \sin\theta_i. \tag{91}$$

Substituting (91) into (89), we obtain

$$f_i'' - f_i(\theta_i')^2 + \frac{1}{R^2}a_i f_i \sum_{j=1}^{N}\frac{1}{a_j}[(f_j')^2 + f_j^2(\theta_j')^2] = 0, \tag{92}$$

$$2f_i'\theta_i' + f_i\theta_i'' = 0. \tag{93}$$

Integrating (93) we find that $f_i^2\theta_i'$ are constants of motion. We set

$$f_i^2\theta_i' = \frac{\sqrt{d_i}}{b_i^2}. \tag{94}$$

and if $f_j = q_j/b_j$ as in the N-dimensional Jacobi system, we finally obtain

$$\frac{d^2 q_i}{ds^2} - \frac{d_i}{q_i^3} + \frac{1}{R^2}a_i q_i \sum_{j=1}^{N}\frac{1}{b_j}\left[\left(\frac{dq_j}{ds}\right)^2 + \frac{d_j}{q_j^2}\right] = 0 \tag{95}$$

with R^2 given by (44).

16

It is interesting to note that N integrals of motion $H_k, (k = 1, ..., N)$, for the Rosochatius system are given by

$$H_k = F_k + F_{k+N} = z_k^2 + z_{k+N}^2 + \frac{1}{z^2} \sum_{l \neq k}^{2N} \frac{J_{kl}^2}{a_k - a_l} + \frac{1}{z^2} \sum_{l \neq k+N}^{2N} \frac{J_{k+N,l}^2}{a_{k+N} - a_l}, \quad (96)$$

where $J_{kl} = z_k w_l - z_l w_k$. It should be noted that both F_k and F_{k+N} contain singular terms because the Hamiltonian (83) is degenerate, $a_k = a_{k+N}, (k = 1, ..., N)$. However, such singular terms cancel in $F_k + F_{k+N}$ and we have

$$H_k = x_k^2 + \frac{1}{x^2} \sum_{l \neq k}^{N} \frac{H_{kl}^2}{a_k - a_l}, \quad (97)$$

with[9,10]

$$H_{kl}^2 = (x_k y_l - x_l y_k)^2 + \frac{c_k x_l^2}{x_k^2} + \frac{c_l x_k^2}{x_l^2}. \quad (98)$$

In a similar way we obtain N integrals $I_k, (k = 1, ..., N)$ for the equations (95) combining pairs of integrals for the $2N$-dimensional degenerate Jacobi system

$$I_k = G_k + G_{k+N} = \pi_k^2 + \pi_{k+N}^2 + \sum_{l \neq k}^{2N} \frac{K_{kl}^2}{b_k - b_l} + \sum_{l \neq k+N}^{2N} \frac{K_{k+N,l}^2}{b_{k+N} - b_l}, \quad (99)$$

where $K_{kl} = \zeta_k \pi_l - \zeta_l \pi_k$ with

$$\begin{aligned}
\zeta_k &= b_k g_k = b_k f_k \cos\theta_k = q_k \cos\theta_k, \\
\zeta_{k+N} &= b_{k+N} g_{k+N} = q_k \sin\theta_k, \\
\pi_k &= \zeta_k' = (p_k \cos\theta_k - q_k \theta_k' \sin\theta_k), \\
\pi_{k+N} &= \zeta_{k+N}' = (p_k \sin\theta_k + q_k \theta_k' \cos\theta_k).
\end{aligned} \quad (100)$$

Substituting (100) into (99), we find

$$I_k = p_k^2 + \frac{d_k}{q_k^2} + \sum_{l \neq k}^{N} \frac{I_{kl}^2}{b_k - b_l} \quad (101)$$

17

with

$$I_{kl}^2 = (q_k p_l - q_l p_k)^2 + \frac{d_k q_l^2}{q_k^2} + \frac{d_l q_k^2}{q_l^2}. \tag{102}$$

Since it can be shown that

$$\frac{dI_k}{ds} = 0, \quad (k = 1, ..., N), \tag{103}$$

I_k are indeed conserved quantities of the system governed by (95).

5. Summary and concluding remarks

In the first part of this work we dealt with the Neumann and the Jacobi systems in the classical framework of constrained Hamiltonian systems. We calculated Dirac brackets for canonical variables. We noticed that a transmutation from the second-class constraints to the first-class ones occurred by changing dynamical variables appropriately. As a consequence, both systems acquired gauge freedom in terms of residual gauge-parameters.

We focused on the relationship between the Neumann and the Jacobi systems. The mapping from the phase space of the Jacobi system to that of the Neumann system was executed by the Gauss map. Affine connections appearing in the geodesic equations in the Jacobi system are given by

$$\Gamma_{jk}^i = \frac{1}{R^2} \frac{q_i}{b_i} \frac{1}{b_j} \delta_{jk}. \tag{104}$$

However, these affine connections do not satisfy the identities which genuine affine connections should satisfy, because the geodesic equations are not all independent.

In the second part of the work we considered the Rosochatius system. We derived the Hamiltonian $\mathcal{H}_R$ for the Rosochatius system from the $2N$-dimensional Neumann Hamiltonian. The geodesic equations for the Rosochatius system should be obtained by making use of the $2N$-dimensional Jacobi equations. However, the

final result given by (95) is by no means the geodesic equation for q_i, because (95) does not have the standard form of the geodesic equation, which is usually written as

$$\frac{d^2 f_k}{ds^2} + \Gamma_{ij}^k \frac{df_i}{ds} \frac{df_j}{ds} = 0,$$

$$\Gamma_{ij}^k = \frac{\phi_k}{\sum_{l=1}^{N} \phi_l^2} \sum_{i,j=1}^{N} \phi_{ij}, \tag{105}$$

where ϕ is the regular function of f_i, $(i = 1, ..., N)$, $\phi_i = \partial \phi / \partial f_i$ and $\phi_{ij} = \partial^2 \phi / \partial f_i \partial f_j$.

However, we have

$$\frac{d^2 \theta_k}{ds^2} + \frac{2}{q_k} \frac{dq_k}{ds} \frac{d\theta_k}{ds} = 0,$$

$$\frac{d^2 q_k}{ds^2} - q_k \left(\frac{d\theta_k}{ds}\right)^2 + \frac{1}{R^2} a_k q_k \sum_{j=1}^{N} \frac{1}{b_j} \left[\left(\frac{dq_j}{ds}\right)^2 + q_j^2 \left(\frac{d\theta_j}{ds}\right)^2\right] = 0 \tag{106}$$

from (93),(94) and (95). These equations are geodesic equations in the $2N$-dimensional space described by the coordinates (q_k, θ_k), $(k = 1, ..., N)$.

We found that (95) are integrable and I_k, $(k = 1, ..., N)$ are conserved quantities of the system governed by (95). In other words, the set of equations given by (95) constitutes an integrable system dual to the Rosochatius system.

There are some important problems on the Rosochatius system with its dual system left unsolved, such as finding classical solutions in arbitrary dimensions and the problem of quantizing the Jacobi-Neumann and the Rosochatius systems. Some classical solutions to the Neumann system have been found by several authors [6]. Quantization of the Neumann system is discussed in some extent by Gurarie [14]. We would like to discuss extensively those remaining problems in a forthcoming paper.

Acknowledgement

References

[1] Neumann C 1859 *J. Reine Angew. Math.* **56** 46-63

[2] Jacobi C G J 1884 *Vorlesungen über Dynamik* in *Gesammelte Werke Supplementband* (Berlin)

[3] Moser J 1979 *Geometry of Quadrics and Spectral Theory*(Chern Symposium)(Berkeley) 147-188;
Moser J 1978 *Prog. Math.***8** 233-289;
Moser J 1981 Integral Hamiltonian Systems and Spectral Theory *Pisa,Lezioni Fermiane*

[4] Avan J and Talon M 1991*Phys.Lett.***B268** 209-216;
Avan J and Talon M 1990 *Int.J.Mod.Phys.* **A5** 4477-4488;
Avan J and Talon M 1991 *Nucl.Phys.***B352** 215-249
Avan J, *Phys.Lett.***A156** 61-68
Babelon O and Talon M 1992 *Nucl. Phys.* **B379** 321-339

[5] Ragnisco O and Suris Y B 1996 On the r-Matrix Structure of the Neumann System and its Discretizations, in *Algebraic Aspects of Integrable Systems* ed Fokas A S and Gelfand I M *Prog.Nonlinear Diff.Eqs. and Their Applications* (Birkhäuser)
Ratiu T (1981) *Trans. Amer. Math. Soc.* **264** 321-329

[6] Mumford D 1983 *Tata Lectures on Theta II* (Birkhaeuser)
Dubrovin B A 1981 *Russ. Math. Surv.* **36** 11-80
Semenov-Tian-Shansky M A 1994 *Integrable Systems II* in *Dynamical Systems VII* ed Arnol'd V I and Novikov S P (Verlin: Springer-Verlag)

Veselov A P 1980 *Funct. Anal. Appl.* **14** 37-39

Devaney R L 1976 *Amer. J. Math.* **100** 631-642

[7] Knörrer H 1982 *J. Reine Angew. Math.* **334** 69-78.

[8] Adler M and van Moerbeke P 1988 *Commun. Math. Phys.* **113** 659-700.

[9] Rosochatius E 1877 *Über die Bewegung eines Punktes*, (Inaugural Dissertation, Universität Göttingen, Gebr. Unger) (Berlin)

[10] Ratiu T 1982 The Lie Algebraic Interpretation of the Complete Integrability of the Rosochatius System in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, AIP Conference Procs.***88** 109-115 (AIP, New York)

Gagnon L, Harnad J and Winternitz P 1985 *J. Math. Phys.* **26** 1605-1612

[11] Uhlenbeck K K 1982 *Lecture Notes in Mathematics* **949** 146-158

[12] Dirac P A M 1950 *Can. J. Math.* **2** 129-148;
Dirac P A M 1951 *Can. J. Math.* **3** 1-23;
Dirac P A M 1958 *Proc. Roy. Soc.* **A246** 326-338;
Dirac P A M 1967 *Lectures on Quantum Mechanics*
(New York: Yeshiva University, Academic Press)

[13] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton: Princeton University Press)
Sundermeyer K 1982 *Constrained Dynamics, Lecture Notes in Physics* **169** (Berlin, Springer)
Hanson A, Regge T and Teitelboim C 1976 *Constrained Hamiltonian Systems   Accad. Naz. dei Lincei* (Rome)
Kugo T 1989 *The Quantum Theory of Gauge Fields* I, II, (Tokyo: Baifukan), (in Japanese), and references therein.

[14] Gurarie D 1995 *J. Math. Phys.* **36** 5355-5391